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FIXED POINTS FOR SOME CONTRACTIVE MAPPING IN PARTIAL METRIC SPACES

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Abstract. Matthews introduced the concepts of partial metric spaces and proved the Banach fixed point theorem in complete partial metric spaces. Dukic, Kadelburg, and Radenovic proved fixed point theorems for Geraghty-type mappings in complete partial metric spaces. In this paper, we prove the fixed point theorem for some contractive mapping in a complete partial metric space.

1. Introduction and Preliminaries

Metric spaces has been generalized in many ways. Among others, the notion of a partial metric space was introduced in 1992 by Matthews [5] to model computation over a metric space. His goal was to study the reality of finding closer and closer approximation to a given number and showing that contractive algorithms would serve to find these approximations.

Definition 1.1. Let X be a non-empty set. Then a mapping $d : X \times X \longrightarrow [0, \infty)$ is called *a partial metric* if for any $x, y, z \in X$, the following conditions hold:
(pm1) $d(x, x) \leq d(x, y)$,
(pm2) $d(x, y) = d(y, x)$,
(pm3) if $d(x, x) = d(x, y) = d(y, y)$, then $x = y$, and
(pm4) $d(x, z) + d(y, y) \leq d(x, y) + d(y, z)$.
In this case, (X, d) is called *a partial metric space.*

Example 1.2.
(1) *Let* $X = [0, \infty)$ *and* $d(x, y) = \max\{x, y\}$ *for all* $x, y \in X$. *Then* (X, d) *is a partial metric space.*

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(2) *Let* $X = \{[a,b] | a, b \in \mathbb{R}, a \leq b\}$ *and* $d([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$. *Then* (X,d) *is a partial metric space.*

Let (X,d) be a partial metric space. For any $x \in X$ and $\epsilon > 0$, let

$$B_d(x,\epsilon) = \{y \in X \mid d(x,y) - d(x,x) < \epsilon\}.$$

Lemma 1.3. [6] *Let* (X,d) *be a partial metric space. Then we have the followings:*
(1) $\{B_d(x,\epsilon) \mid x \in X, \epsilon > 0\}$ *is a base for some topology* τ_d,
(2) (X,τ_d) *is a* T_0*-space, and*
(3) *a sequence* $\{x_n\}$ *converges to* x *in* (X,τ_d) *if and only if* $\lim_{n\to\infty} d(x_n,x) = d(x,x)$.

Let (X,d) be a partial metric space. A sequence $\{x_n\}$ in (X,d) is called *Cauchy* if $\lim_{n,m\to\infty} d(x_m,x_n)$ exists and is finite and (X,d) is called *complete* if every Cauchy sequence $\{x_n\}$ in (X,d) converges to x in (X,τ_d) such that

$$\lim_{n\to\infty} d(x_n,x) = d(x,x) = \lim_{n,m\to\infty} d(x_m,x_n).$$

Lemma 1.4. [8] *Let* (X,d) *be a partial metric space. Then a sequence* $\{x_n\}$ *converges to* x *in* (X,τ_d) *with* $d(x,x) = 0$ *if and only if for any* $y \in X$, $\lim_{n\to\infty} d(x_n,y) = d(x,y)$.

There exist many generalizations of the well-known Banach contraction mapping principle in the literature. In particular, Matthews [5], [6] proved the Banach fixed point theorem in partial metric spaces and after that, fixed point results in partial metric spaces have been studied by many authors([1], [3], [7]).

First, the well-known Banach contraction theorem [2] is stated as follows.

Theorem 1.5. *Let* (X,d) *be a complete metric space and let* $f : X \longrightarrow X$ *a contraction mapping, that is, there exists* $\lambda \in [0,1)$ *such that*

$$p(Tx,Ty) \leq \lambda p(x,y)$$

for all $x, y \in X$. *Then* f *has a unique fixed point.*

Matthews [6] proved the Banach fixed point theorem in partial metric spaces as follows:

Theorem 1.6. [6] *Let* (X,d) *be a complete partial metric space and let* $f : X \longrightarrow X$ *a contraction mapping, that is, there exists* $\lambda \in [0,1)$ *such that*

$$p(Tx,Ty) \leq \lambda p(x,y)$$

for all $x, y \in X$. *Then* f *has a unique fixed point* $u \in X$ *with* $p(u, u) = 0$.

Now, let Σ be the set of all functions $\beta : [0, \infty) \longrightarrow [0, 1)$ which satisfies

$$\lim_{n\to\infty} \beta(t_n) = 1 \textit{ implies} \lim_{n\to\infty} t_n = 0.$$

Dukic, Kadelburg, and Radenovic [4] proved the following fixed point theorem for Geraghty contractions :

THEOREM 1.7. [4] *Let be a complete partial metric space and* $f : X \longrightarrow X$ *a mapping. Suppose that there is a* $\beta \in \Sigma$ *such that*

$$d(f(x), f(y)) \leq \beta(d(x, y))d(x, y) \tag{1.1}$$

for all $x, y \in X$. *Then* f *has a unique fixed point* $u \in X$ *with* $d(u, u) = 0$.

Recently, Altun and Sadarangani [1] proved the following fixed point theorem for Geraghty contractions :

THEOREM 1.8. [1] *Let* (X, d) *be a partial complete metric space. Suppose that* $f : X \longrightarrow X$ *is a mapping such that there is a* $\beta \in \Sigma$ *with*

$$d(f(x), f(y)) \leq \beta(A(x, y)) \max\{d(x, y), d(x, fx), d(y, fy)\}$$

for all $x, y \in X$, *where*

$$A(x, y) = \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{1}{2}[d(x, fy) + d(fx, y)]\right\}.$$

Then f *has a unique fixed point* u *in* X.

In this paper, we will prove a fixed point theorem for some contractive mapping in a complete partial metric space.

2. Fixed point theorem for some contractive mapping in a complete partial metric space

Now, we will prove a fixed point theorem for some contractive mapping in complete partial metric spaces.

THEOREM 2.1. *Let* (X, d) *be a complete partial metric space and* $f : X \longrightarrow X$ *a mapping such that there is a* $\beta \in \Sigma$ *such that*

$$d(f(x), f(y)) \leq \beta(A(x, y))A(x, y) \tag{2.1}$$

for all $x, y \in X$, *where*

$$A(x, y) = \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{1}{2}[d(x, fy) + d(fx, y)]\right\}.$$

Then there is a unique fixed point u *of* f *with* $d(u, u) = 0$.

Proof. Let $x \in X$ and for any $n \in \mathbb{N}$, let $f^{n+1}x = ff^nx$ and $x_n = f^nx$.

Suppose that $d(x_m, x_{m+1}) = 0$ for some $m \in \mathbb{N}$. Then we have $x_m = x_{m+1}$, that is, x_m is a fixed point of f and $d(x_m, x_m) = 0$. Hence one has the results.

Suppose that $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, let $\alpha_n = d(x_n, x_{n+1})$. By (2.1),

$$d(x_n, x_n) \leq \beta(A(x_{n-1}, x_{n-1}))A(x_{n-1}, x_{n-1}) \leq \beta(A(x_{n-1}, x_{n-1}))\alpha_{n-1}$$

for all $n \in \mathbb{N}$. Then by (2.1),

$$\begin{aligned} &\alpha_{n+1} \\ &\leq \beta(A(x_{n+1}, x_n))A(x_{n+1}, x_n) \\ &\leq \beta(A(x_{n+1}, x_n)) \max\Big\{\alpha_n, \alpha_{n+1}, \frac{1}{2}\Big[d(x_{n+1}, x_{n+1}) + d(x_{n+2}, x_n)\Big]\Big\} \end{aligned} \tag{2.2}$$

for all $n \in \mathbb{N}$. If $\max\Big\{\alpha_n, \alpha_{n+1}, \frac{1}{2}\Big[d(x_{n+1}, x_{n+1}) + d(x_{n+2}, x_n)\Big]\Big\} = \alpha_{n+1}$ for some $n \in \mathbb{N}$, then by (2.2), we have $\alpha_{n+1} = 0$, because $\alpha_{n+1} \neq 0$ and $0 \leq \beta(A(x_{n+1}, x_n)) < 1$, which is a contradiction. Hence we have

$$\alpha_{n+1} < \max\Big\{\alpha_n, \alpha_{n+1}, \frac{1}{2}\Big[d(x_{n+1}, x_{n+1}) + d(x_{n+2}, x_n)\Big]\Big\} \tag{2.3}$$

for all $n \in \mathbb{N}$. Henc by (2.2) and (2.3), we get

$$\alpha_{n+1} \leq \beta(A(x_{n+1}, x_n)) \max\Big\{\alpha_n, \frac{1}{2}\Big[d(x_{n+1}, x_{n+1}) + d(x_{n+2}, x_n)\Big]\Big\} \tag{2.4}$$

for all $n \in \mathbb{N}$.

Suppose that there is an $n \in \mathbb{N}$ such that

$$\alpha_n \leq \frac{1}{2}\Big[d(x_{n+1}, x_{n+1}) + d(x_{n+2}, x_n)\Big]. \tag{2.5}$$

Then by (2.5) and (pm4), we have

$$\begin{aligned} \alpha_n &\leq \frac{1}{2}\Big[d(x_{n+1}, x_{n+1}) + d(x_{n+2}, x_n)\Big] \\ &\leq \frac{1}{2}\alpha_{n+1} + \frac{1}{2}\alpha_n \end{aligned} \tag{2.6}$$

and so

$$\alpha_n \leq \alpha_{n+1}. \tag{2.7}$$

By (2.6) and (2.7), we have

$$\frac{1}{2}\Big[d(x_{n+1}, x_{n+1}) + d(x_{n+2}, x_n)\Big] \leq \frac{1}{2}\alpha_{n+1} + \frac{1}{2}\alpha_n \leq \alpha_{n+1}$$

and thus

$$\alpha_{n+1} = \max\Big\{\alpha_n, \frac{1}{2}\Big[d(x_{n+1}, x_{n+1}) + d(x_{n+2}, x_n)\Big]\Big\},$$

which is a contradiction. Hence we have

$$\alpha_n > \frac{1}{2}\Big[d(x_{n+1}, x_{n+1}) + d(x_{n+2}, x_n)\Big]. \tag{2.8}$$

for all $n \in \mathbb{N}$. Thus by (2.4) and (2.8), we have

$$\alpha_{n+1} \leq \beta(A(x_{n+1}, x_n))\alpha_n < \alpha_n \tag{2.9}$$

for all $n \in \mathbb{N}$ and so $\{\alpha_n\}$ is a bounded below real decreasing sequence. Thus there is a non-negative real number α with $\lim_{n\to\infty} \alpha_n = \alpha$.

Suppose that $\alpha > 0$. Letting $n \to \infty$ in (2.9), we get $\lim_{n\to\infty} \beta(A(x_{n+1}, x_n)) = 1$. Since $\beta \in \Sigma$ and $A(x_{n+1}, x_n) = \alpha_n$,

$$0 = \lim_{n\to\infty} A(x_{n+1}, x_n) = \lim_{n\to\infty} \alpha_n = \alpha,$$

which is a contradiction. Thus $\alpha = 0$ and so

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.10}$$

Now, we claim that $\{x_n\}$ is a Cauchy sequence in (X, d). Enough to show that $\lim_{n,m\to\infty} d(x_m, x_n) = 0$. Suppose that $\lim_{n,m\to\infty} d(x_m, x_n) \neq 0$. Then there is an $\epsilon > 0$ and there are subsequences $\{x_{m(k)}\}$, $\{x_{n(k)}\}$ of $\{x_n\}$ such that $m(k) > n(k) > k$ and

$$d(x_{m(k)}, x_{n(k)}) \geq \epsilon \tag{2.11}$$

for all $k \in \mathbb{N}$. Moreover, for any $k \in \mathbb{N}$, we can choose $m(k)$ in such a way that it is smallest integer with $m(k) > n(k)$ and satisfies (2.11). Then

$$d(x_{m(k)-1}, x_{n(k)}) < \epsilon \tag{2.12}$$

and by (2.11) and (2.12), we have

$$\begin{aligned}\epsilon &\leq d(x_{m(k)}, x_{n(k)})\\ &\leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)}) - d(x_{m(k)-1}, x_{m(k)-1})\\ &\leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)})\\ &< d(x_{m(k)}, x_{m(k)-1}) + \epsilon\end{aligned} \tag{2.13}$$

for all $k \in \mathbb{N}$. By (2.10) and (2.13), we have

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon \tag{2.14}$$

and

$$\begin{aligned} &d(x_{m(k)-1}, x_{n(k)-1}) \\ &\le d(x_{m(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)-1}) \end{aligned}$$

for all $k \in \mathbb{N}$. Letting $k \to \infty$ in the last inequality, by (2.10), we get

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) \le \epsilon. \tag{2.15}$$

Since

$$\begin{aligned} &d(x_{m(k)}, x_{n(k)}) \\ &\le d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) \end{aligned}$$

for all $k \in \mathbb{N}$,

$$\epsilon \le \lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}). \tag{2.16}$$

By (2.15) and (2.16), we have

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)-1}) = \epsilon. \tag{2.17}$$

By (2.10) and (2.12), we have

$$\begin{aligned} \epsilon &\le d(x_{m(k)}, x_{n(k)}) \le \beta(A(x_{m(k)-1}, x_{n(k)-1}))A(x_{m(k)-1}, x_{n(k)-1}) \\ &\le \beta(A(x_{m(k)-1}, x_{n(k)-1})) \max\Big\{d(x_{m(k)-1}, x_{n(k)-1}), d(x_{m(k)-1}, x_{m(k)}), \\ &\quad d(x_{n(k)}, x_{n(k)-1}), \frac{1}{2}[d(x_{m(k)-1}, x_{n(k)}) + d(x_{m(k)}, x_{n(k)-1})]\Big\} \\ &\le \beta(A(x_{m(k)-1}, x_{n(k)-1})) \max\Big\{d(x_{m(k)-1}, x_{n(k)-1}), d(x_{m(k)-1}, x_{m(k)}), \\ &\quad d(x_{n(k)}, x_{n(k)-1}), \frac{1}{2}[\epsilon + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)}, x_{n(k)-1})]\Big\} \end{aligned}$$

and by (2.14) and (2.17), we get

$$\lim_{k\to\infty} \beta(A(x_{m(k)-1}, x_{n(k)-1})) = 1.$$

Hence $\lim_{k\to\infty} A(x_{m(k)-1}, x_{n(k)-1}) = 0$ and so

$$\lim_{n\to\infty} d(x_{m(k)}, x_{n(k)}) = 0$$

which is a contradiction to (2.11). Thus

$$\lim_{n,m\to\infty} d(x_m, x_n) = 0 \tag{2.18}$$

and so $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete partial metric space, there is an u in X such that

$$\lim_{n\to\infty} d(x_n, u) = d(u, u) = \lim_{n,m\to\infty} d(x_n, x_m).$$

By (2.18), we get

$$\lim_{n\to\infty} d(x_n, u) = d(u, u) = 0. \tag{2.19}$$

By Lemma 1.4 and (2.19), we have

$$\lim_{n\to\infty} d(x_n, fu) = d(u, fu) = 0 \tag{2.20}$$

and by (pm1) and (2.20),

$$d(u, u) = d(u, fu) = d(fu, fu).$$

Hence $fu = u$ and thus u is a fixed point of f.

To prove the uniqueness of u, let v be another fixed point of f with $d(v, v) = 0$. Then we have

$$d(u, v) = d(fu, fv) \le \beta(A(u, v))A(u, v) = \beta(A(u, v))d(u, v).$$

Since $0 \le \beta(A(u, v)) < 1$, $u = v$. □

Using Theorem 2.1, we have the following corollary :

COROLLARY 2.2. *Let (X, d) be a partial complete metric space and $f : X \longrightarrow X$ a mapping . Suppose that there is a $\beta \in \Sigma$ with*

$$d(f(x), f(y)) \le \beta(A(x, y))B(x, y)$$

for all $x, y \in X$, where

$$B(x, y) = d(x, y),\ B(x, y) = \max\{d(x, y), d(x, fx), d(y, fy)\},$$

$$\textit{or}\ \ B(x, y) = \frac{1}{2}[d(x, fy) + d(fx, y)]$$

Then f has a unique fixed point $u \in X$.

In particular, if $B = \max\{d(x, y), d(x, fx), d(y, fy)\}$ in Corollary 2.2, its result is Theorem 1.8.

EXAMPLE 2.3. *Let $X = [0, 1]$ and define a pratial metric $d : X \times X \longrightarrow [0, \infty)$ by $d(x, y) = \max\{x, y\}$. Then (x, d) is a complete partial metric space. Define a mapping $f : X \longrightarrow X$ by $f(x) = \frac{1}{3}$ and define a mapping $\beta : [0, \infty) \longrightarrow [0, 1)$ by*

$$\beta(t) = \begin{cases} \frac{10}{10+t}e^{-t}, & \textit{if } t > 0, \\ \frac{1}{2}, & \textit{if } t = 0. \end{cases}$$

Then clearly, $\beta \in \Sigma$. Since $d(f0, f1) = \frac{1}{3} > \frac{10}{11e} = \beta(d(0, 1))d(0, 1)$, (1.1) in Theorem 1.7 does not satisfied.

For any $x, y \in X$ *with* $x \geq y$ *and* $x > 0$, $A(x, y) = x$ *and since* $\beta(t) \geq \frac{10}{11e}$ *for all* $t \in X$,

$$\beta(A(x,y))A(x,y) = \beta(x)x = \frac{10x}{(10+x)e^x} \geq x > \frac{x}{3} = d(fx, fy).$$

Hence all conditions of Theorem 2.1 are satisfied and thus f *has the unique fixed point* $u \in X$ *with* $d(u, u) = 0$.

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